Model Answer

AS-2827

B.Sc. (Hon's) Fifth Semester, Examination, 2013

Mathematics

Linear Algebra

Ans. 1. (a). In $\mathbb{R}(\mathbb{Z}_2)$ distributive ~~property~~ condition does not holds because

$$(\bar{1} \oplus \bar{1}) \cdot 1 = \bar{2} \cdot 1 = \bar{0} \cdot 1 = 0$$

and $\quad \bar{1} \cdot 1 + \bar{1} \cdot 1 = 1 + 1 = 2$

therefore $\quad (\bar{1} + \bar{1}) \cdot 1 \neq \bar{1} \cdot 1 + \bar{1} \cdot 1$

1(b). Linear sum of two subspaces: Let W_1 and W_2 be two subspaces of a vector space $V(F)$. Then linear sum of W_1 and W_2 is defined as

$$W_1 + W_2 = \{ w_1 + w_2 \; ; \; w_1 \in W_1 \text{ and } w_2 \in W_2 \}$$

1(c). Linearly independent set of vectors:

Let $S = \{a_1, a_2, \cdots a_n\}$ be a non-empty set of vectors which is subset of a vector space $V(F)$. Then S is called linearly independent if

$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = 0$ holds only for all α's $= 0_F$.

eg. $\{(1,0), (0,1)\}$ is Linearly independent subset in $\mathbb{R}^2$.

1(d). Subspace generated by a subset:

Let S be a subset of a vector space $V(F)$. Then the smallest subspace of V containing S, is called as subspace generated by S. It is denoted by $[S]$.

1(e). Fundamental theorem of homomorphism:

A homomorphic image of a vector space, is isomorphic to some quotient space of the vector space.

1(f). Inner Product space: Let V be a vector space over the field F of complex number. Then a map $\langle\ \rangle : V \times V \to F$ is called an inner product in V if

(I) $\langle a+b, c\rangle = \langle a, c\rangle + \langle b, c\rangle \quad , \forall a, b, c \in V$

(II) $\langle \lambda a, b\rangle = \lambda \langle a, b\rangle \quad , \forall \lambda \in F, \ a, b \in V$

(III) $\langle a, b\rangle = \overline{\langle b, a\rangle} \quad , \forall a, b \in V$

(IV) $\langle a, a\rangle \geq 0_F \quad , \forall a \in V$

equality holds iff $a = 0$

The vector space V with this inner product $\langle\ \rangle$ is called an inner product space $(V, \langle\ \rangle)$.

1(g). Linear functional: A linear transformation $f : V(F) \to F(F)$ is called a linear functional on the vector space $V(F)$.

1(h). $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

2. Necessary and sufficient condition for a non-empty subset W of a vector space to be a subspace. ~~is~~ of vector space $V(F)$ is

$a, b \in W$ & $\alpha, \beta \in F \Rightarrow \alpha \cdot a + \beta \cdot b \in W$

Proof. Necessary Part- Let W be a subspace of the vector space $V(F)$, then V is closed with respect to + and $\cdot$ and therefore

$a, b \in W$ & $\alpha, \beta \in F \Rightarrow \alpha \cdot a + \beta \cdot b \in W$.

Sufficient Condition: Let the condition holds, then we have to show that W is a subspace of V(F).

(I) (W,+) is a commutative group. In order to prove that (W,+) is a commutative group, it is sufficient to prove that (W,+) is a subgroup of (V,+).

Let $a, b \in W$. Take $\alpha = 1,\ \beta = -1$

then by given condition

$$1 \cdot a + (-1) \cdot b \in W$$

$$\text{ie.} \quad a + (-1) \cdot b \in W$$

$$\text{ie.} \quad a - b \in W$$

therefore (W,+) is a subgroup of (V,+).

$$\left.\begin{array}{ll} \text{(II)(i)} & \alpha \cdot (\beta \cdot a) = (\alpha \odot \beta) \cdot a \\ \text{(ii)} & 1 \cdot a = a \\ \text{(III)(i)} & \alpha \cdot (a+b) = \alpha \cdot a + \alpha \cdot b \\ \text{(ii)} & (\alpha \oplus \beta) \cdot a = \alpha \cdot a + \beta \cdot a \end{array}\right\} \begin{array}{l} \forall\ \alpha, \beta \in F,\ a, b \in W \\ \text{holds as } W \text{ is a subset} \\ \text{of } V. \end{array}$$

Sol^n. 3. Let $a_1, a_2, \cdots, a_n$ be linearly dependent vectors of a vector space V(F). Then

$$\alpha_1 a_1 + \alpha_2 a_2 + \text{-----} + \alpha_n a_n = 0 \quad \text{where at least one } \alpha \neq 0_F \quad \text{--(1)}$$

without loss of generality, we may assume that $\alpha_1 \neq 0_F$

then $\exists\ \alpha_1^{-1} \in F$ s.t. $\alpha_1^{-1} \odot \alpha_1 = 1$

From eq^n (1), we get

$$\lambda_1 a_1 = (-\lambda_2 a_2) + (-\lambda_3 a_3) + \text{---} + (-\lambda_n a_n)$$

$$\lambda_1^{-1} \cdot (\lambda_1 a_1) = \lambda_1^{-1}(-\lambda_2 a_2) + \lambda_1^{-1}(-\lambda_3 a_3) + \text{---} + \lambda_1^{-1}(-\lambda_n a_n)$$

$$(\lambda_1^{-1} \odot \lambda_1)\, a_1 = (-\lambda_1^{-1} \odot \lambda_2) a_2 + (-\lambda_1^{-1} \odot \lambda_3) a_3 + \text{---} + (-\lambda_1^{-1} \odot \lambda_n) a_n$$

$$1 \cdot a_1 = \mu_2 a_2 + \mu_3 a_3 + \text{---} + \mu_n a_n \quad \text{where } \mu_i = -\lambda_1^{-1} \odot \lambda_i$$

ie. a_1 is linear combination of $a_2, a_3, \ldots, a_n$.

Conversely, let one of the vectors, a_1 can be written as linear combination of remaining vectors $a_2, a_3, \ldots, a_n$.

ie. $a_1 = \lambda_2 a_2 + \cdots + \lambda_n a_n$; λ's $\in F$

$$(-1) \cdot a_1 + \lambda_2 \cdot a_2 + \cdots + \lambda_n a_n = 0$$

Since $-1 \neq 0_F$

therefore $a_1, a_2, \ldots a_n$ are linearly dependent vectors.

Sol^n. 4: Let $V_1 \cong V_2$ then $\exists$ an isomorphism $T: V_1 \to V_2$.

Then $\operatorname{Ker} T = \{0_1\}$ and $\operatorname{Im} T = V_2$

therefore $\dim \operatorname{Ker} T = 0$ and $\dim \operatorname{Im} T = \dim V_2$

By Rank-Nullity Theorem,

$$\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \dim V_1$$

$$0 + \dim V_2 = \dim V_1$$

ie. $\dim V_1 = \dim V_2$

Conversely,

Suppose $\dim V_1 = \dim V_2 = n$ (say)

Let $X = \{x_1, x_2, \ldots, x_n\}$ & $Y = \{y_1, y_2, \ldots, y_n\}$ be basis of V_1 and V_2 respectively. Let $a, b \in V_1$ s.t.

$$a = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

$$b = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n$$ where α's and β's $\in F$

Let us define $T: V_1 \to V_2$ by

$$T(a) = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n \quad \forall a \in V_1$$

T is well-defined.

Let $a = b$

$$\alpha_1 x_1 + \dots + \alpha_n x_n = \beta_1 x_1 + \dots + \beta_n x_n$$

$$(\alpha_1 - \beta_1) x_1 + \dots + (\alpha_n - \beta_n) x_n = 0$$

Since $\{x_1, x_2, \dots x_n\}$ is linearly independent,

$$\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \dots = \alpha_n - \beta_n = 0_F$$

then $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\dots$, $\alpha_n = \beta_n$

therefore $\alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n = \beta_1 y_1 + \dots + \beta_n y_n$

ie $T(a) = T(b)$

T is linear transformation.

Let $a, b \in V$ and $\lambda, \mu \in F$

$$T(\lambda a + \mu b) = T\left[\lambda(\alpha_1 x_1 + \dots + \alpha_n x_n) + \mu(\beta_1 x_1 + \dots + \beta_n x_n)\right]$$

$$= T\left[(\lambda\alpha_1 \oplus \mu\beta_1) x_1 + \dots + (\lambda\alpha_n \oplus \mu\beta_n) x_n\right]$$

$$= (\lambda\alpha_1 \oplus \mu\beta_1) y_1 + \dots + (\lambda\alpha_n \oplus \mu\beta_n) y_n$$

$$= \lambda\left[\alpha_1 y_1 + \dots + \alpha_n y_n\right] + \mu\left[\beta_1 y_1 + \dots + \beta_n y_n\right]$$

$$= \lambda\left[T(a)\right] + \mu\left[T(b)\right]$$

$\therefore$ T is a linear transformation

T is one - one.

Let $T(a) = T(b)$

$$\alpha_1 y_1 + \dots + \alpha_n y_n = \beta_1 y_1 + \dots + \beta_n y_n$$

$$(\alpha_1 - \beta_1) y_1 + \dots + (\alpha_n - \beta_n) y_n = 0$$

$\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \dots = \alpha_n - \beta_n = 0_F$ being $\{y_1, y_2, \dots y_n\}$ lin. indep.

$\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\dots$, $\alpha_n = \beta_n$

therefore $\alpha_1 x_1 + \dots + \alpha_n x_n = \beta_1 x_1 + \dots \beta_n x_n$

ie. $a = b$

T is onto. Since $\dim V_1 = \dim V_2$, therefore T is also onto.

Hence T is isomorphism.

Therefore $V_1 \cong V_2$.

Sol^n 5: Let $T_1, T_2 : V_1 \to V_2$ be two linear transformations. then $T_1 + T_2 : V_1 \to V_2$ is also a linear transformation.

We will first show that $Im(T_1+T_2) \subseteq Im\,T_1 + Im\,T_2$

Let $(T_1+T_2)(x) \in Im(T_1+T_2)$

Since $(T_1+T_2)(x) = \underset{\in Im\,T_1}{T_1(x)} + \underset{\in Im\,T_2}{T_2(x)}$

therefore $Im(T_1+T_2) \subseteq Im\,T_1 + Im\,T_2$ ----(1)

Since $Im(T_1+T_2)$ is a subspace of V_2 and linear sum of $Im\,T_1$ and $Im\,T_2$ is also a subspace of V_2. Therefore from (1) we get that $Im(T_1+T_2)$ is a subspace of $Im\,T_1 + Im\,T_2$.

Hence
$$\begin{aligned} \dim Im(T_1+T_2) &\leq \dim(Im\,T_1 + Im\,T_2) \\ &= \dim Im\,T_1 + \dim Im\,T_2 - \dim(Im\,T_1 \cap Im\,T_2) \\ &\leq \dim Im\,T_1 + \dim Im\,T_2 \end{aligned}$$

ie. $\text{Rank}(T_1+T_2) \leq \text{Rank}\,T_1 + \text{Rank}\,T_2$

Sol^n 6. A linear transformation $T: V_1 \to V_2$ is said to be non-singular if T is one-one onto.

Now we will show that composition of two linear trans. is also a linear transformation. Let $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ be two linear transformations. Then $T_2 \circ T_1 : V_1 \to V_3$ is defined as
$$(T_2 \circ T_1)(x) = T_2[T_1(x)]$$

Let $a, b \in V_1$ and $\alpha, \beta \in F$

$$\begin{aligned} (T_2 \circ T_1)(\alpha a + \beta b) &= T_2[T_1(\alpha a + \beta b)] \\ &= T_2[\alpha \cdot T_1(a) + \beta \cdot T_1(b)] && \text{as } T_1 \text{ is lin. trans.} \\ &= \alpha \cdot T_2[T_1(a)] + \beta \cdot T_2[T_1(b)] && \text{as } T_2 \text{ is lin. trans.} \\ &= \alpha (T_2 \circ T_1)(a) + \beta (T_2 \circ T_1)(b) \end{aligned}$$

Therefore $T_2 \circ T_1 : V_1 \to V_3$ is a linear trans.

Also since the composition of two bijective maps is a bijective map.

Hence the composition of two non-singular linear transformation is also a linear transformation.

Conversely, let $T_1 : \mathbb{R} \to \mathbb{R}^2$ be a map defined by

$$T_1(x) = (x, 0) \qquad \forall x \in \mathbb{R}$$

and $T_2 : \mathbb{R}^2 \to \mathbb{R}$ be a map defined by

$$T_2(x, y) = x + y \qquad \forall (x, y) \in \mathbb{R}^2$$

we will first show that T_1 and T_2 are linear trans.

Also $T_2 \circ T_1 : \mathbb{R} \to \mathbb{R}$ be

$$\begin{aligned}(T_2 \circ T_1)(x) &= T_2[T_1(x)] \\ &= T_2[(x, 0)] \\ &= x + 0 = x \\ &= I(x)\end{aligned}$$

Hence $T_2 \circ T_1$ is bijective linear trans. while T_1 is not non-singular being not onto. Also T_2 is not non-singular being not one-one.

Sol^n 7: $X = \{x_1 = (1, -1, 0),\ x_2 = (0, 1, -1),\ x_3 = (0, 3, -2)\}$

Let us define a linear functional $f_i : \mathbb{R}^3 \to \mathbb{R}$ as

$$f_i[(x, y, z)] = a_i x + b_i y + c_i z$$

If f_i is element of dual basis then $f_i(x_j) = \delta_{ij}$

$f_1(x_1) = 1 \Rightarrow f_1(1, -1, 0) = 1$ ie. $a_1 - b_1 = 1$

$f_1(x_2) = 0 \Rightarrow f_1(0, 1, -1) = 0$ ie. $b_1 - c_1 = 0$

$f_1(x_3) = 0 \Rightarrow f_1(0, 3, -2) = 0$ ie. $3b_1 - 2c_1 = 0$

from these equations, we get $a_1 = 1,\ b_1 = 0,\ c_1 = 0$.

Also $f_2(x_1) = 0 \Rightarrow f_2(1,-1,0) = 0$ ie. $a_2 - b_2 = 0$

$f_2(x_2) = 0 \Rightarrow f_2(0,1,-1) = 1$ ie. $b_2 - c_2 = 1$

$f_2(x_3) = 0 \Rightarrow f_2(0,3,-2) = 0$ ie. $3b_2 - 2c_2 = 0$

from these equations, we get $a_2 = -2$, $b_2 = -2$, $c_2 = -3$

Again $f_3(x_1) = 0 \Rightarrow f_3(1,-1,0) = 0$ ie. $a_3 - b_3 = 0$

$f_3(x_2) = 0 \Rightarrow f_3(0,1,-1) = 0$ ie. $b_3 - c_3 = 0$

$f_3(x_3) = 1 \Rightarrow f_3(0,3,-2) = 1$ ie. $3b_3 - 2c_3 = 1$

from these equations, we get $a_3 = 1$, $b_3 = 1$, $c_3 = 1$

Hence the dual basis is $X^* = \{f_1, f_2, f_3\}$

Sol^n Q:

$S = \{b_1 = (1,0,0),\ b_2 = (1,1,0),\ b_3 = (1,1,1)\}$

a_1 is given by $a_1 = \frac{b_1}{\|b_1\|} = \frac{(1,0,0)}{\sqrt{1^2+0^2+0^2}} = (1,0,0)$

$c_2 = b_2 - \langle b_2, a_1 \rangle a_1$

$= (1,1,0) - 1(1,0,0)$

$= (0,1,0)$

$\therefore a_2 = \frac{c_2}{\|c_2\|} = \frac{(0,1,0)}{\sqrt{0^2+1^2+0^2}} = (0,1,0)$

$c_3 = b_3 - \langle b_3, a_1 \rangle a_1 - \langle b_3, a_2 \rangle a_2$

$= (1,1,1) - 1(1,0,0) - 1(0,1,0)$

$= (0,0,1)$

$\therefore a_3 = \frac{c_3}{\|c_3\|} = \frac{(0,0,1)}{\sqrt{0^2+0^2+1^2}} = (0,0,1)$

Thus the orthonormal set is

$S_1 = \{a_1, a_2, a_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$.

M gupta Ranadive